Numerical Analysis and Programming

Example

Problem Statement:

Use the Taylor series expansions with n=0 to 6 to approximate f(x)=cos x at $x_{i+1} = \pi/3$ on the basis of the value of f(x) and its derivatives at $x_i = \pi/4$. Note that this means that $h = \pi/3 - \pi/4 = \pi/12$

Calculate the percent relative error in each iteration

Roots Of Equations

Root Finding Problems

Many problems in Science and Engineering are expressed as:

Given a continuous function $f(x)$,
find the value r such that $f(r) = 0$

These problems are called root finding problems.

Roots of Equations

A number r that satisfies an equation is called a root of the equation.

The equation : $x^4 - 3x^3 - 7x^2 + 15x = -18$

has four roots : $-2,\ 3,\ 3,\ and\ -1$.

i.e., $x^4 - 3x^3 - 7x^2 + 15x + 18 = (x+2)(x-3)^2(x+1)$

The equation has two simple roots $(-1\ \ and\ \ -2)$

and a repeated root (3) *with multiplicity* $= 2$.

Zeros of a Function

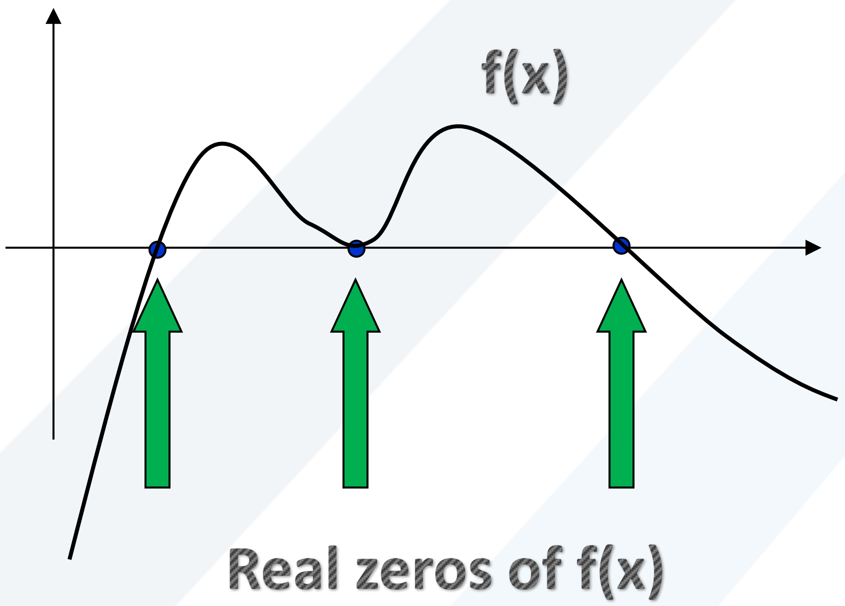
Let **f(x)** be a real-valued function of a real variable. Any number **r** for which **f(r)=0** is called a zero of the function.

Examples:

2 and 3 are zeros of the function **f(x) = (x-2)(x-3)**

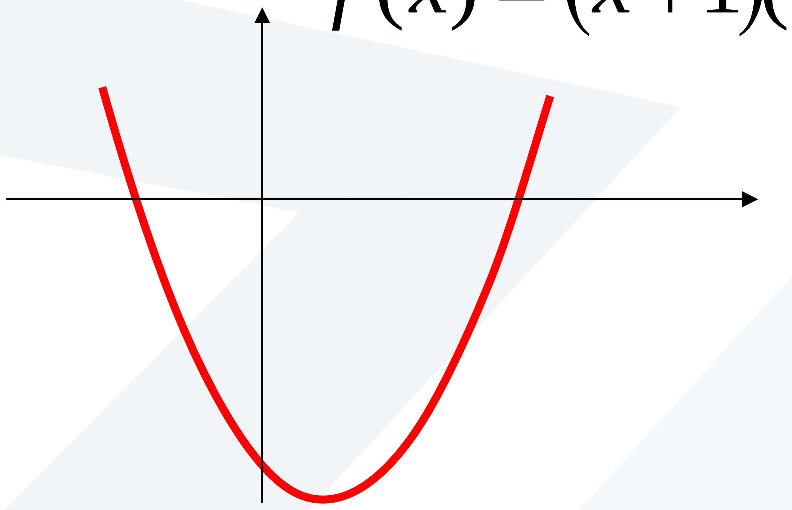
Graphical Interpretation of Zeros

The real zeros of a function $f(x)$ are the values of x at which the graph of the function crosses (or touches) the x-axis.



Simple Zeros

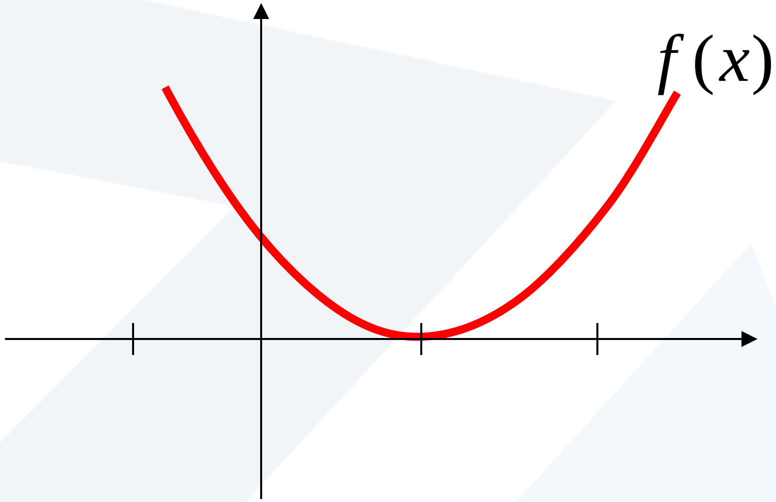
$$f(x) = (x+1)(x-2)$$



$$f(x) = (x+1)(x-2) = x^2 - x - 2$$

has two simple zeros (one at $x = 2$ and one at $x = -1$)

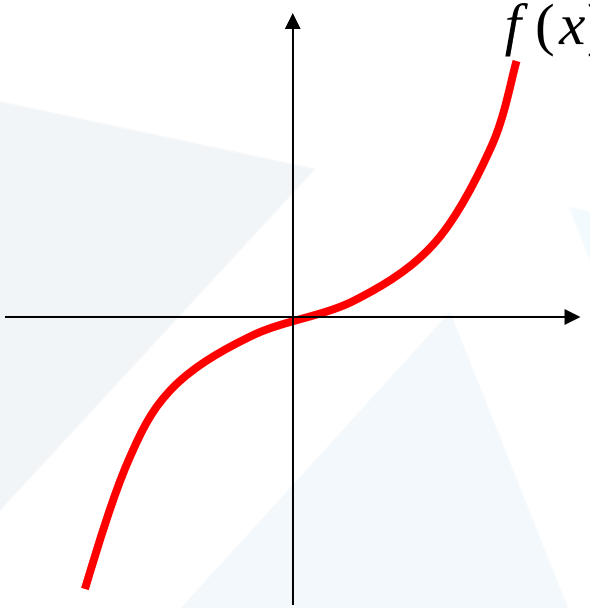
Multiple Zeros



$$f(x) = (x-1)^2 = x^2 - 2x + 1$$

has double zeros (zero with muliplicit y = 2) at x = 1

Multiple Zeros



$$f(x) = x^3$$

has a zero with muliplicity $= 3$ at $x = 0$

Facts

- Any **n**th order polynomial has exactly **n** zeros (counting real and complex zeros with their multiplicities).
- Any polynomial with an odd order has at least one real zero.
- If a function has a zero at **x=r** with multiplicity **m** then the function and its first **(m-1)** derivatives are zero at **x=r** and the **m**th derivative at **r** is not zero.

Roots of Equations & Zeros of Function

Given the equation :

$$x^4 - 3x^3 - 7x^2 + 15x = -18$$

Move all terms to one side of the equation :

$$x^4 - 3x^3 - 7x^2 + 15x + 18 = 0$$

Define $f(x)$ as :

$$f(x) = x^4 - 3x^3 - 7x^2 + 15x + 18$$

The zeros of $f(x)$ are the same as the roots of the equation $f(x) = 0$

(Which are -2, 3, 3, and -1)

Nonlinear Equation Solvers

- Graphical Solutions
- Numerical Solutions
  - Bracketing Methods
    - Bisection Method
    - False Position Method
  - Open Methods
    - Fixed Point Iteration
    - Newton Raphson Method
    - Secant Method
- Analytical Solutions

Analytical Methods

- Analytical Solutions are available for special equations only.
- Analytical Solution for: $ax^2 + bx + c = 0$

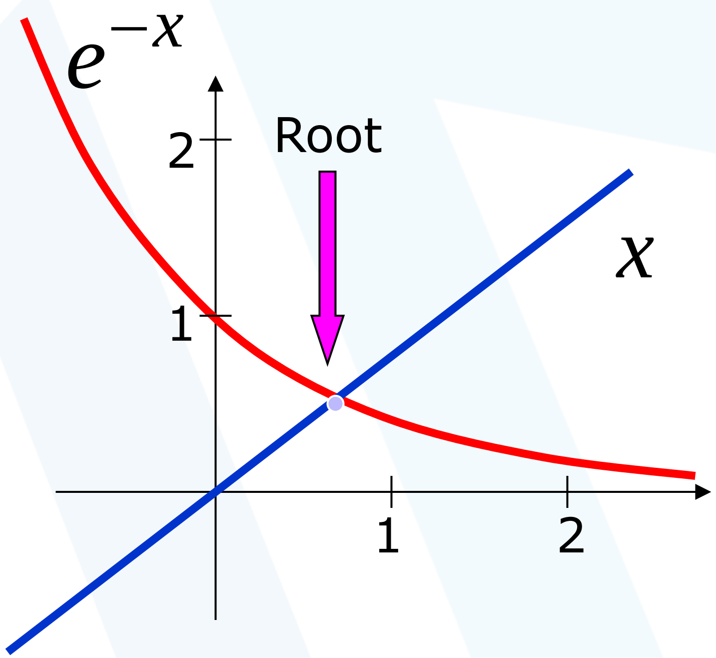
$$roots = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

No analytical solution is available for: $x - e^{-x} = 0$

Graphical Methods

- A simple method for obtaining an estimate of the root of the equation f(x) = 0 is to make a plot of the function and observe where it crosses the x axis. This point, which represents the x value for which f(x) = 0, provides a rough approximation of the root.
- Graphical methods are useful to provide an initial guess to be used by other methods.

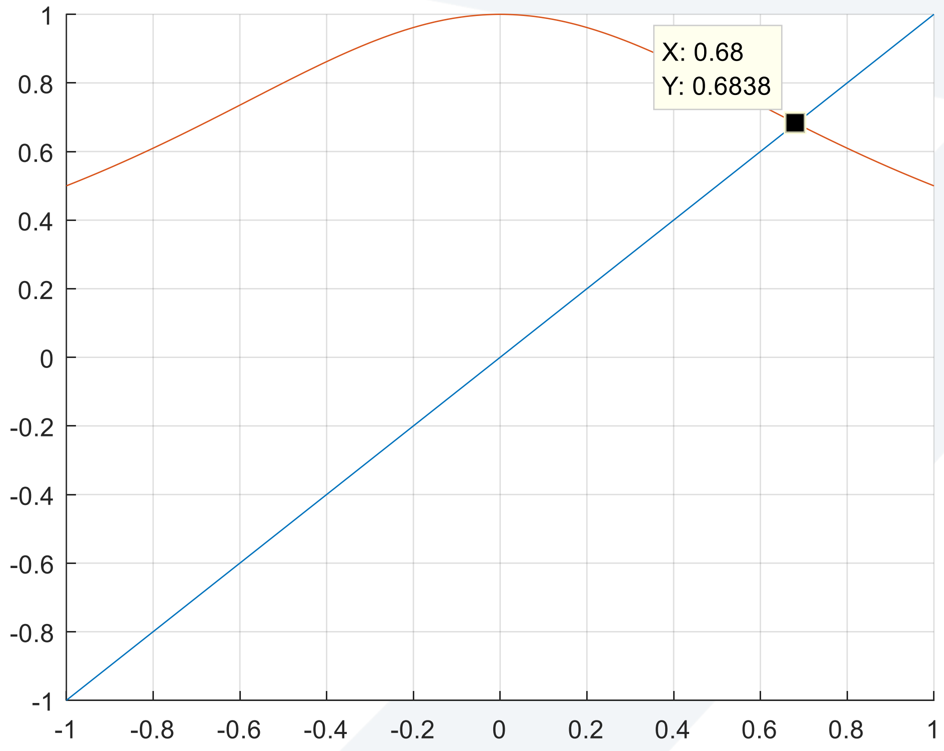
Solve
$x = e^{-x}$
$The\ root \in [0,1]$
$root \approx 0.6$

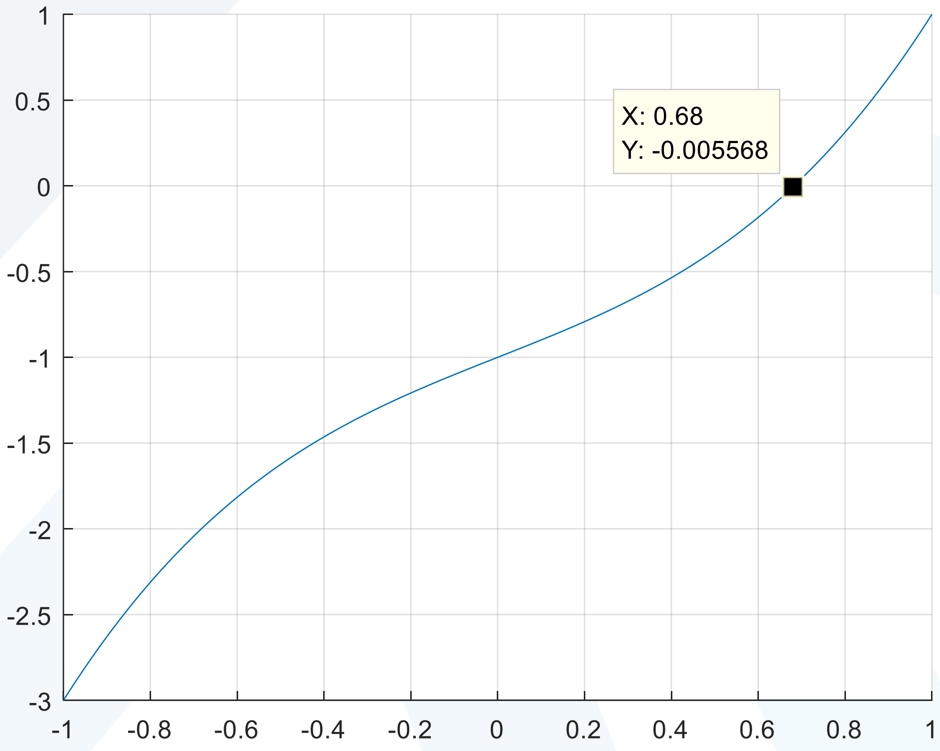


Graphical Methods

$$g(x) = x \quad , \qquad \varphi(x) = \frac{1}{x^2 + 1}$$

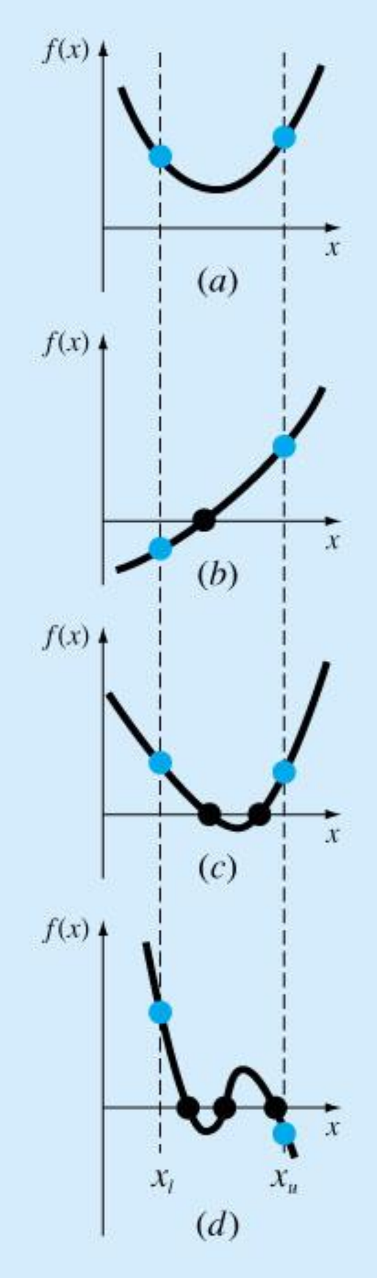
$$f(x) = x^3 + x - 1$$





Graphical Methods

- **Graphical techniques** are of **limited practical value** because they are not precise. However, graphical methods can be utilized to **obtain rough estimates of roots**. These estimates can be employed as **starting guesses for numerical methods** discussed in this and the next lectures. Aside from providing rough estimates of the roots, graphical interpretations are **important tools** for **understanding the properties of the functions** and **anticipating the pitfalls of the numerical methods**. For example, the following shows a number of ways in which roots can occur (or be absent) in an interval prescribed by a lower bound x_l and an upper bound x_u.



No root (same sign)

Single root (change sign)

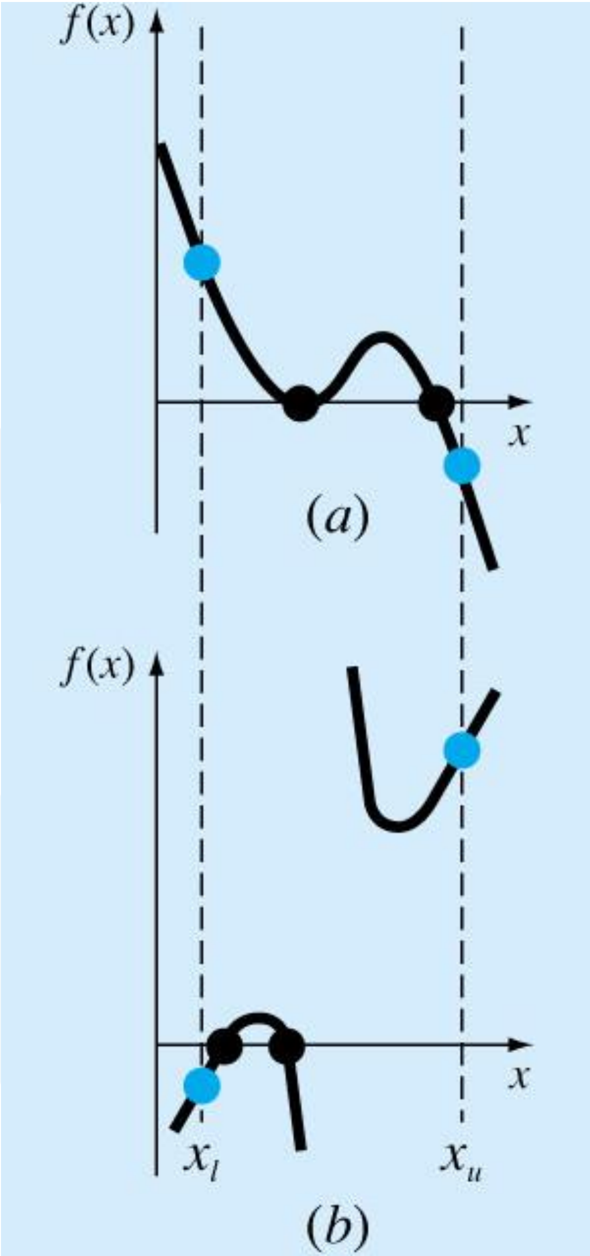
Two roots (same sign)

Three roots (change sign)

Special Cases

Multiple Roots

Discontinuity



Graphical Methods

- **Conclusion:**

Graphical method is useful for getting an idea of what's going on in a problem, but depends on eyeball.

- **Recommendation:**

Use bracketing methods to improve the accuracy

Bracketing Methods

- In bracketing methods (two points method for finding roots), the method starts with an **interval** that contains the root and a procedure is used to obtain a smaller interval containing the root.
- In other words, two initial guesses for the root are required. These guesses must “bracket” or be on either side of the root.
- If one root of a real and continuous function, $f(x)=0$, is bounded by values $x=x_l$, $x=x_u$ then $f(x_l) . f(x_u) < 0$. (The function changes sign on opposite sides of the root)

Bisection Method

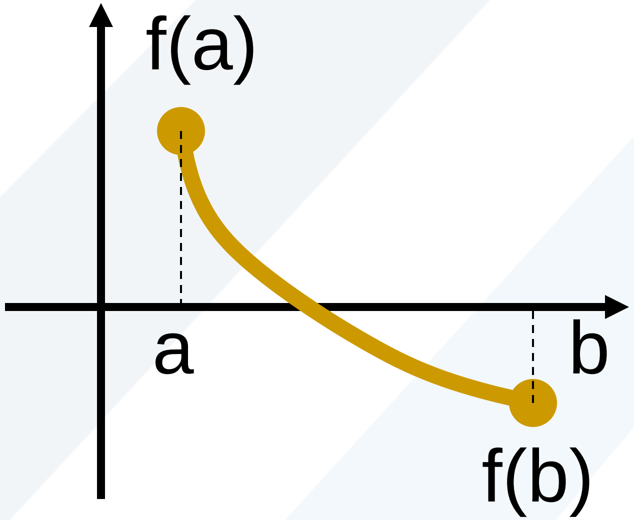
- **Bisection Method:** is one of the simplest methods to find a zero of a nonlinear function.
- It is also called **interval halving** method.
- To use the Bisection method, one needs an **initial interval** that is known to contain a zero of the function.
- The method systematically **reduces** the interval. It does this by dividing the interval into two equal parts, performs a simple test and based on the result of the test, half of the interval is thrown away.
- The procedure is repeated until the desired interval size is obtained.

Bisection Method

- Let f(x) be defined on the interval [a,b].

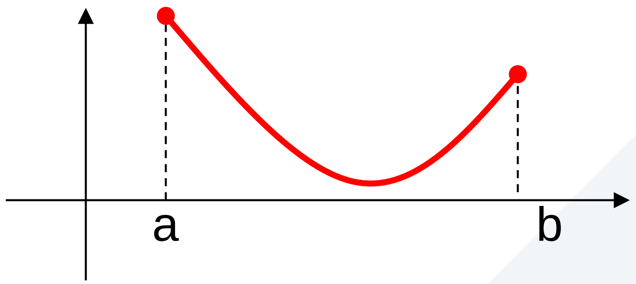
Intermediate value theorem:

if a function is continuous and f(a) and f(b) have different signs then the function has at least one zero in the interval [a,b].

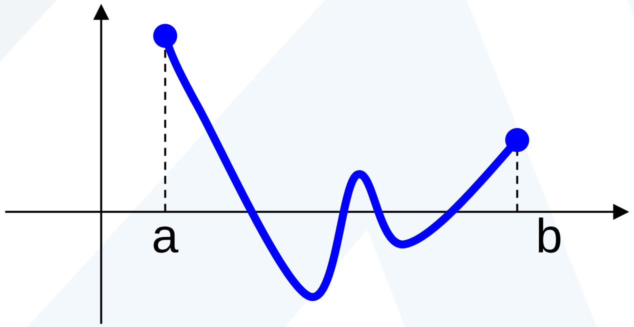


Bisection Method

- If f(a) and f(b) have the same sign, the function may have an even number of real zeros or no real zeros in the interval [a, b].
- Bisection method **can not** be used in these cases.



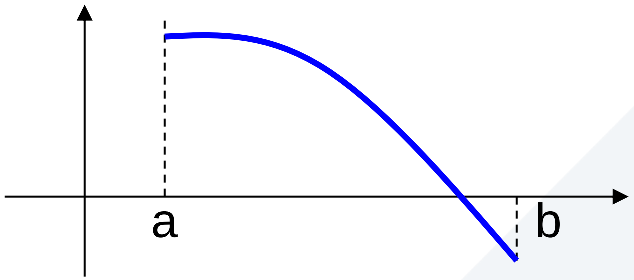
The function has no real zeros



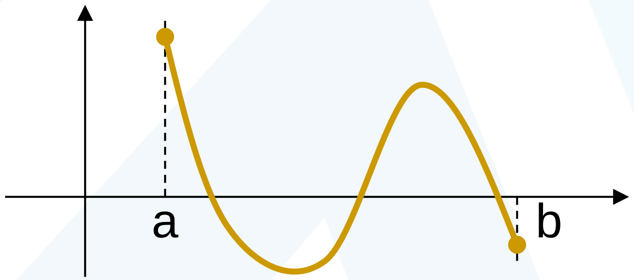
The function has four real zeros

Bisection Method

- If f(a) and f(b) have different signs, the function has at least one real zero.
- Bisection method **can** be used to find one of the zeros.



The function has one real zeros



The function has three real zeros

Bisection Method

- If the function is continuous on [a,b] and f(a) and f(b) have different signs, Bisection method obtains a new interval that is half of the current interval and the sign of the function at the end points of the interval are different.

- This allows us to repeat the Bisection procedure to further reduce the size of the interval.

Bisection Method

Assumptions:

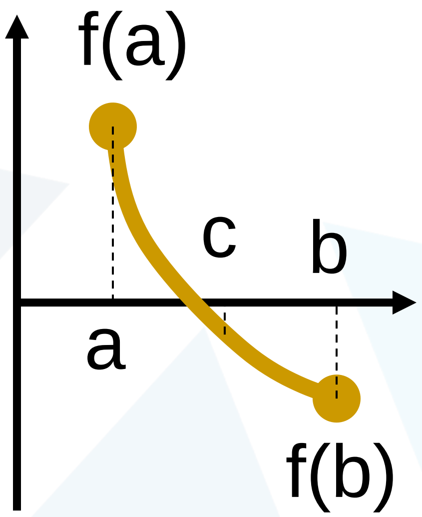
Given an interval [a,b]

f(x) is continuous on [a,b]

f(a) and f(b) have opposite signs.

These assumptions ensure the existence of at least one zero in the interval [a,b] and the bisection method can be used to obtain a smaller interval that contains the zero.

Bisection Algorithm



Assumptions:

$f(x)$ is continuous on $[a,b]$

$f(a) \times f(b) < 0$

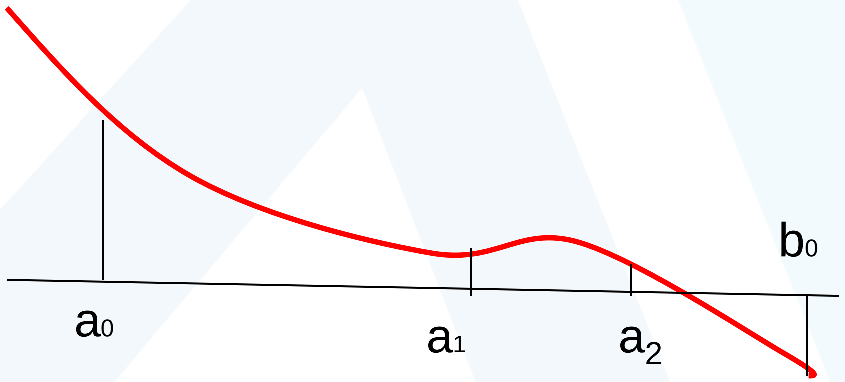
Algorithm:

Loop

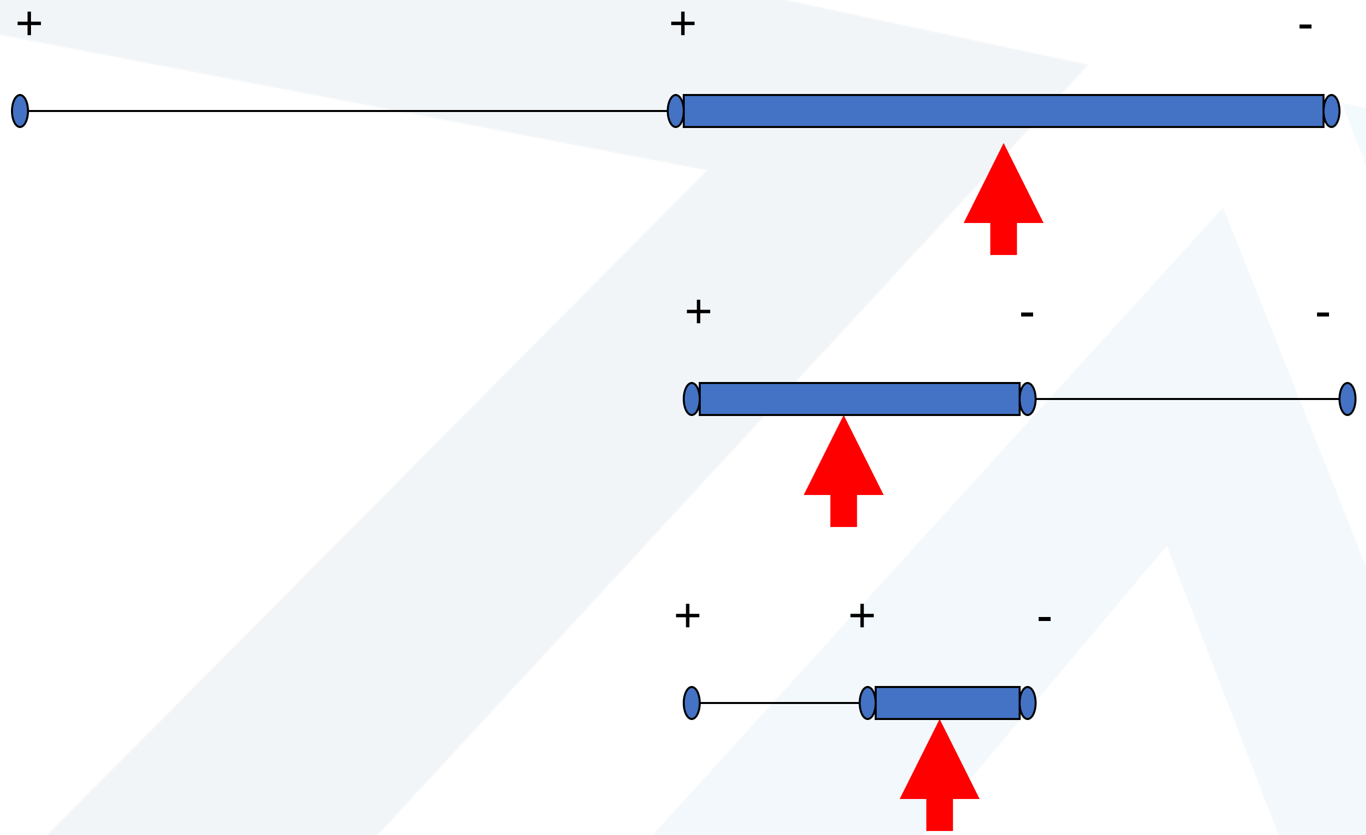
1. Compute the mid point $c=(a+b)/2$
2. Evaluate $f(c)$
3. If $f(a) \times f(c) < 0$ then new interval $[a, c]$

   If $f(a) \times f(c) > 0$ then new interval $[c, b]$

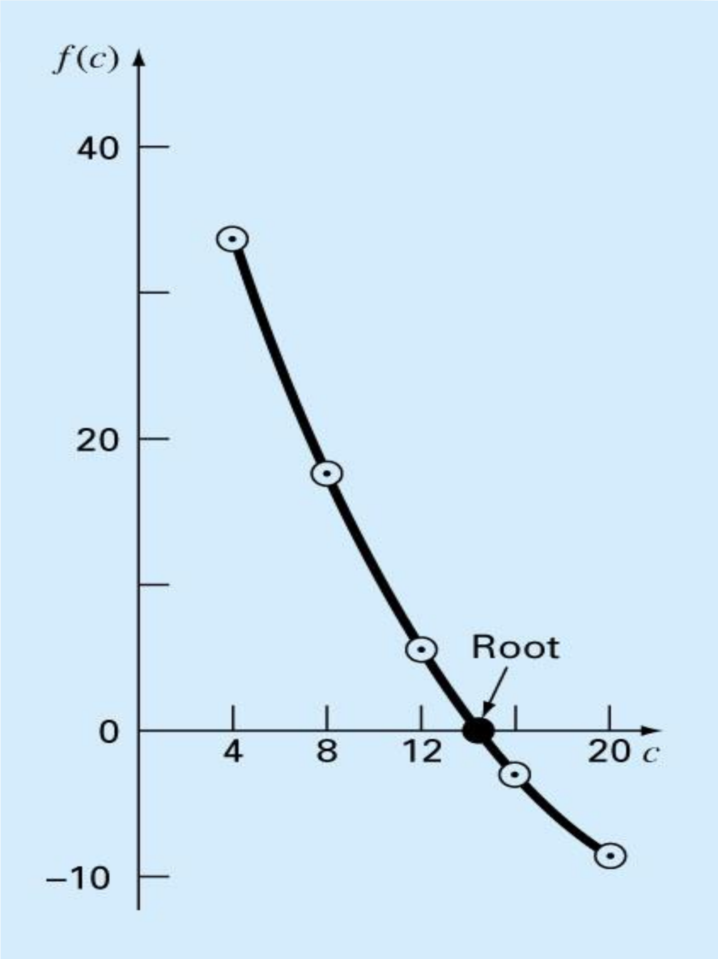
End loop

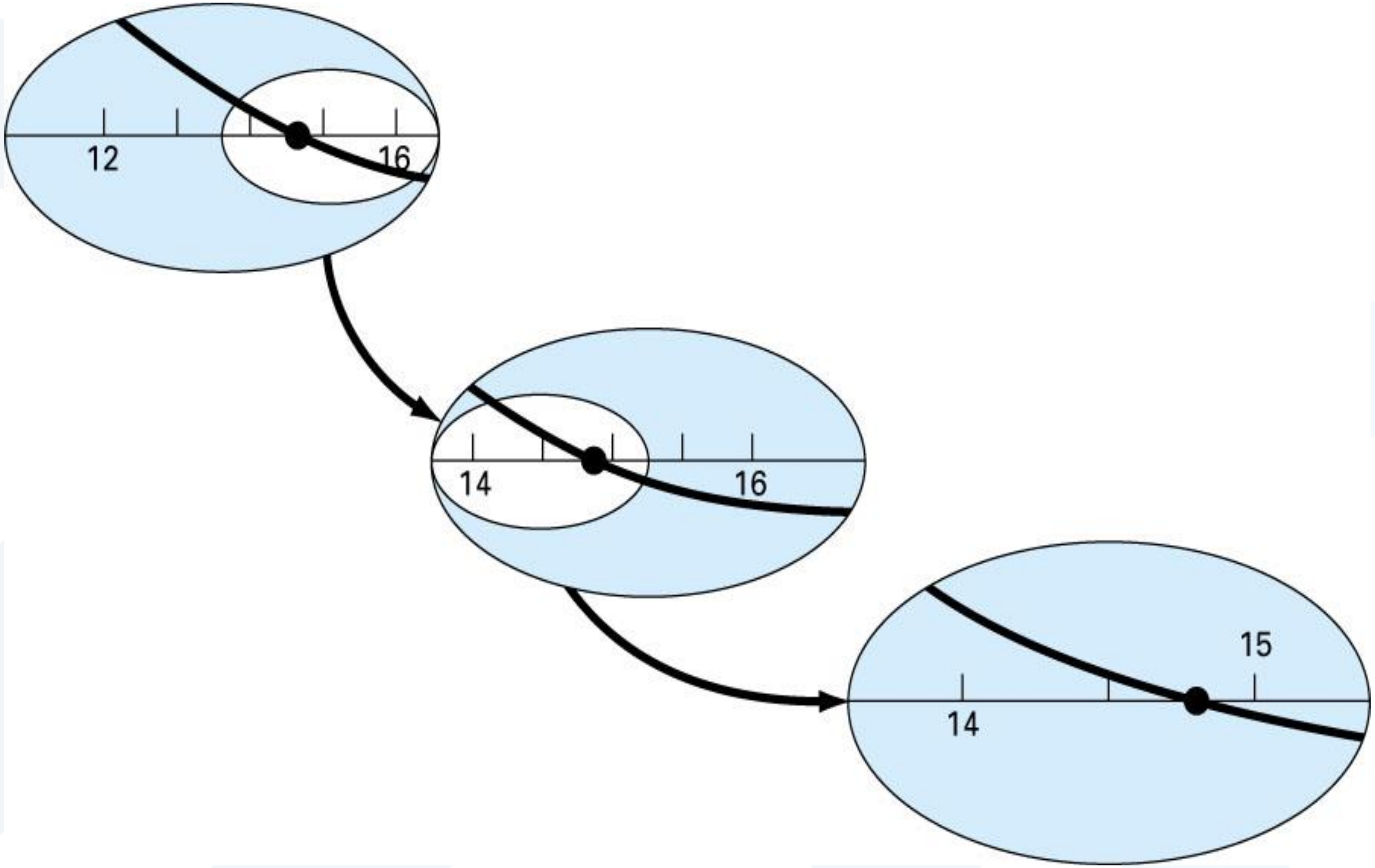


Bisection Method

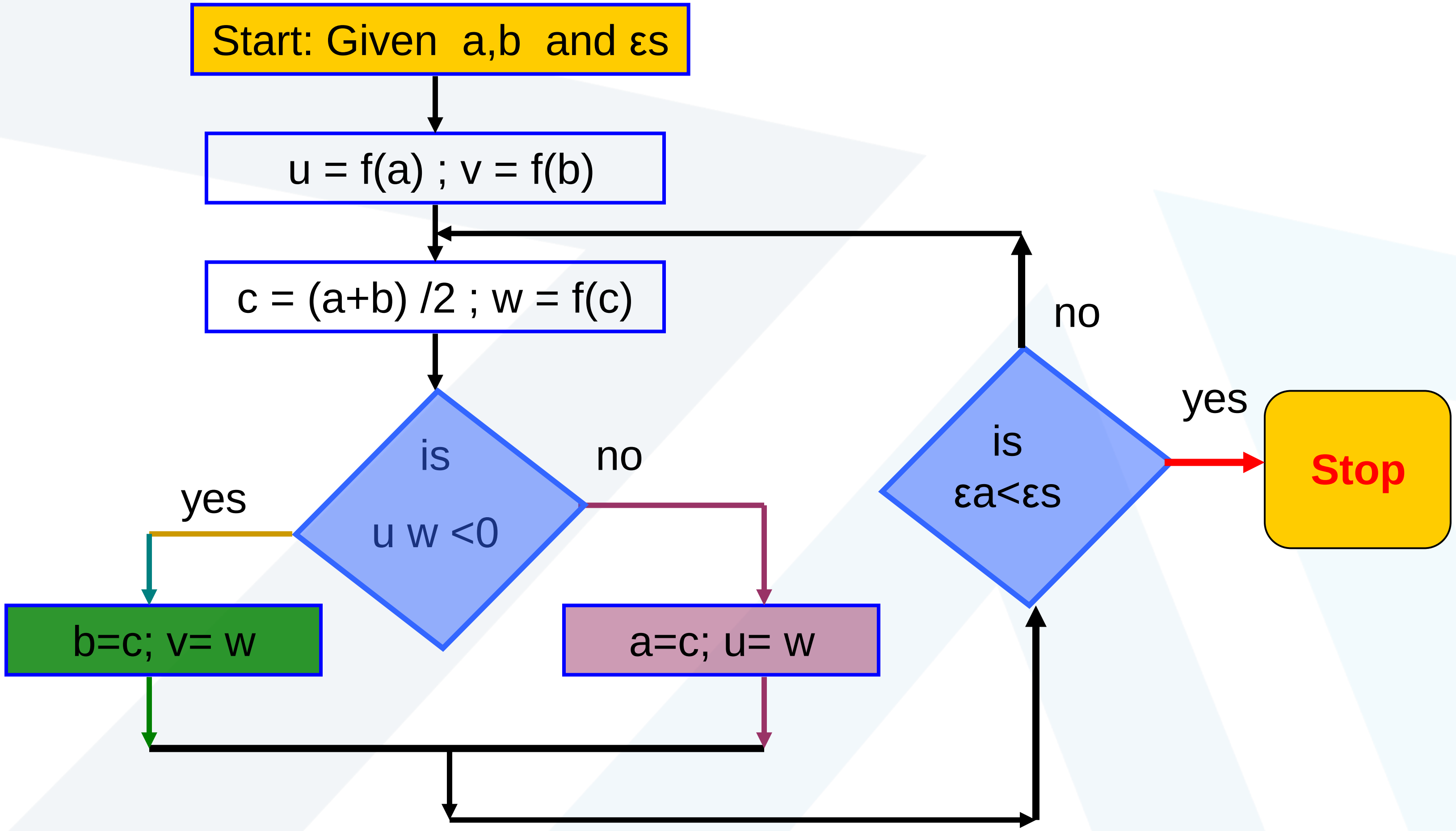


Bisection Method (Examples)





Flow Chart of Bisection Method



Bisection Method (Examples)

Can you use Bisection method to find a zero of :

$f(x) = x^3 - 3x + 1$ in the interval $[0,2]$?

Answer:

$f(x)$ is continuous on $[0,2]$

and $f(0)*f(2) = (1)(3) = 3 > 0$

$\Rightarrow$ Assumptions are not satisfied

$\Rightarrow$ Bisection method can not be used

Bisection Method (Examples)

Can you use Bisection method to find a zero of :

$f(x) = x^3 - 3x + 1$ in the interval $[0,1]$?

Answer:

$f(x)$ is continuous on $[0,1]$

and $f(0)*f(1) = (1)(-1) = -1 < 0$

$\Rightarrow$ Assumptions are satisfied

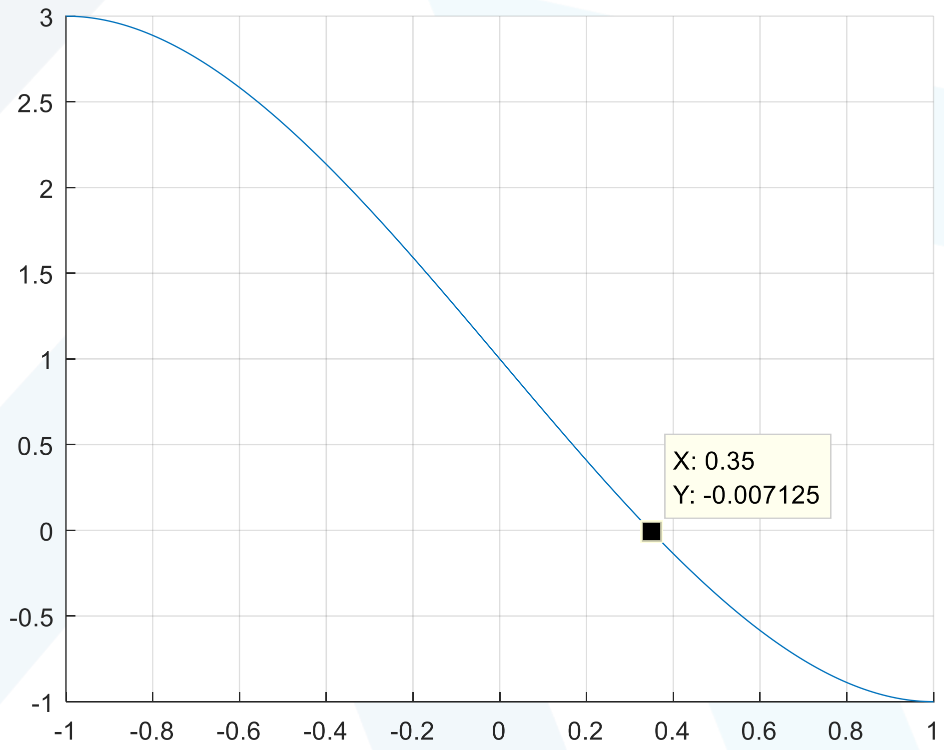
$\Rightarrow$ Bisection method can be used

Bisection Method (Examples)

- **Problem Statement**. Use bisection to solve the same problem approached graphically in slide 16. interval [0,1].

$$f(x) = x^3 - 3x + 1$$

Iteration	a	b	c= $\frac{(a+b)}{2}$	f(c)	



Bisection Method (Error Estimation)

- we require an error estimate that is not contingent on foreknowledge of the root. As developed previously, an approximate percent relative error Ɛa can be calculated, as in

$$\varepsilon_a = \left| \frac{x_r^{new} - x_r^{old}}{x_r^{new}} \right| 100\%$$

- where x_{new} r is the root for the present iteration and x_{old} r is the root from the previous iteration. The absolute value is used because we are usually concerned with the magnitude of Ɛa rather than with its sign. When Ɛa becomes less than a prespecified stopping criterion Ɛs, the computation is terminated.

Bisection Method (Error Estimation)

- **Problem Statement**. Continue previous example until the approximate error falls below a stopping criterion of $\varepsilon_s = 2\ \%$.

Bisection Method (Examples)

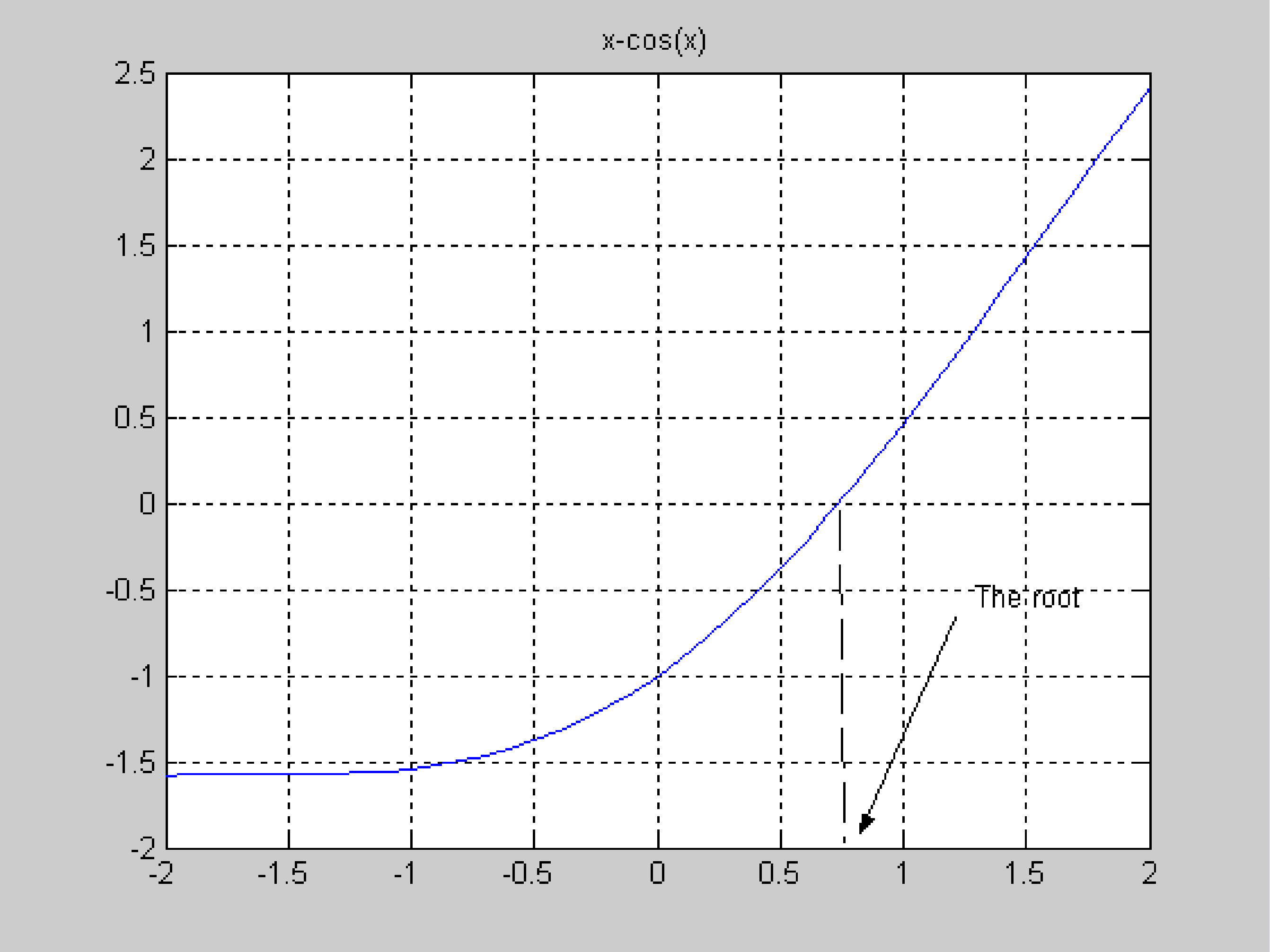
Use Bisection method to find a root of the equation $x = \cos(x)$ with absolute error <0.02 (assume the initial interval [0.5, 0.9])

Question 1: What is f (x) ?

Question 2: Are the assumptions satisfied ?

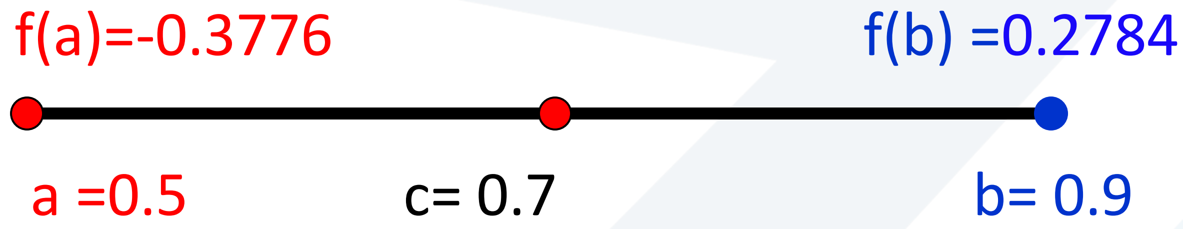
Question 3: How to compute the new estimate ?

Bisection Method (Examples)

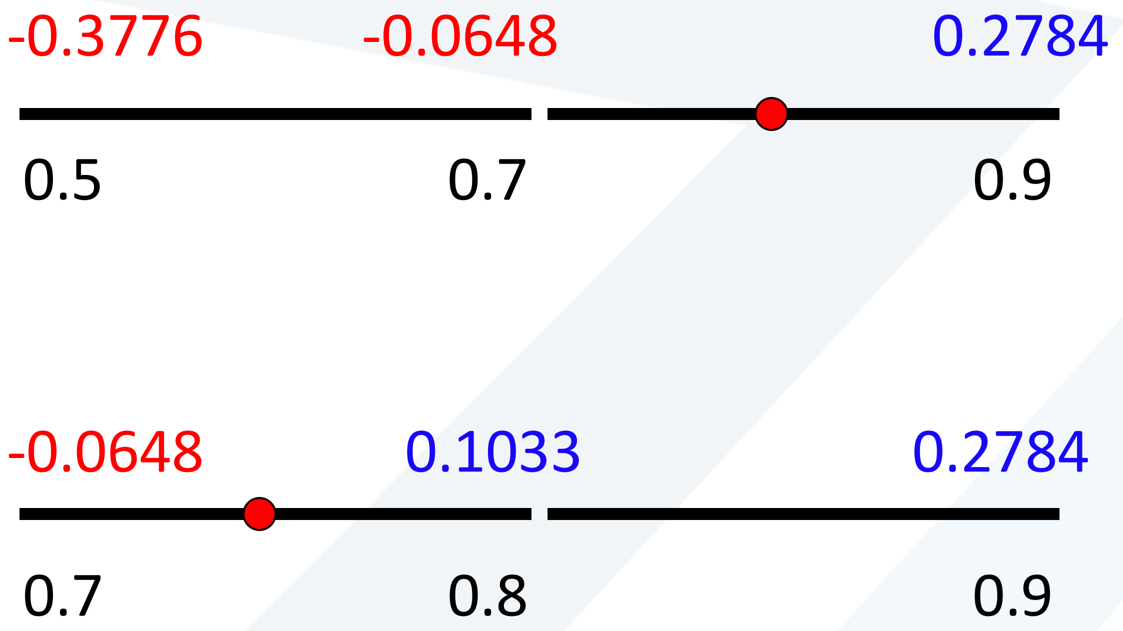


Bisection Method (Examples)

Initial Interval



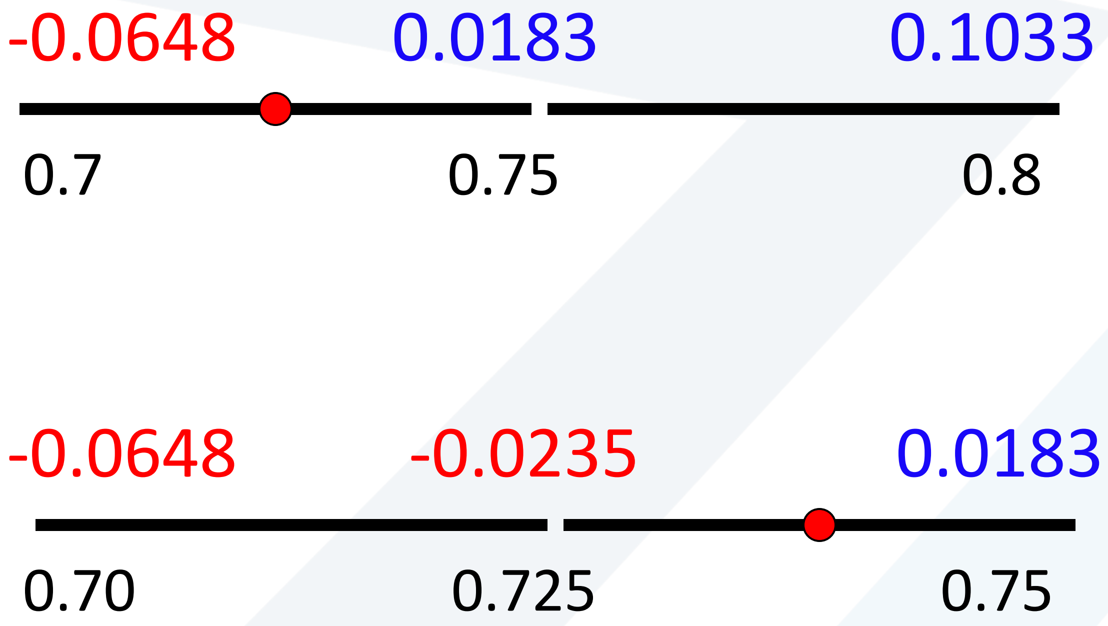
Bisection Method (Examples)



Error =12.5%

Error = 6.66%

Bisection Method (Examples)



Error = 3.45%

Error = 1.75%

Bisection Method (Examples)

Initial interval containing the root: [0.5,0.9]

After 5 iterations:

Interval containing the root: [0.725, 0.75]

Best estimate of the root is 0.7375

$|\text{Error}| < 2\ \%$

Bisection Method

Advantages

- Simple and easy to implement
- One function evaluation per iteration
- The size of the interval containing the zero is reduced by 50% after each iteration
- The number of iterations can be determined a priori
- No knowledge of the derivative is needed
- The function does not have to be differentiable

Disadvantage

- Slow to converge
- Good intermediate approximations may be discarded

Homework

Problem Statement: Determine the real root of :

$$f(x) = -26 + 85x - 91x^2 + 44x^3 - 8x^4 + x^5$$

a) Graphically.

b) Using bisection method to determine the root to ε_s = 10%. Employ initial guess of x_l=0.5 and x_u=1.0.

c) Resolve the previous questions using Excel.